Fixed Points of the q-Bracket on the p-adic Unit Disk

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Abstract. The q-bracket $[X]_q : \mathrm{O}_{\mathbb{C}_p} \to \mathrm{O}_{\mathbb{C}_p}$, which is the q-analog of the identity function, is also a norm-preserving isometry for $q \in 1+S$, where S is the open disk of radius $p^{-1/(p-1)}$ centered at 0. We show $x \in \mathrm{O}_{\mathbb{C}_p}$ is a nontrivial fixed point for $[X]_q$ for some q if and only if $(x-2)\cdots(x-(p-1)) \in S$. The map $Q(X)$ taking x to a q for which $[x]_q = x$ is analytic, and locally $|Q(x')-q| = c\cdot|x'-x|$ for some constant c, unless x is a double (nontrivial) fixed point, in which case $|Q(x')-q| = c\cdot|x'-x|^2$. The nontrivial pairs $(x, q-1)$ such that $[x]_q = x$ form a manifold whose standard projections each have degree $p-2$. Restricting to $\mathbb{Z}_p$, we find the theory to be trivial unless $p=3$, in which case locally $|q'-q| = |x'-x|/3$.

I. Introduction.

Let $\mathbb{C}_p$ denote the p-adic complex numbers. Write $|-|$ for the metric on $\mathbb{C}_p$, and let O_p denote the unit disk in $\mathbb{C}_p$. Write v for the corresponding additive valuation, so that $|x| = p^{-v(x)}$. Let

$$S = \{y \in \mathbb{C}_p : 0 \le |y| < p^{-1/(p-1)}\} \ \subset \mathrm{O}_p$$

If $x \in \mathrm{O}_p$ and $q \in 1+S$ then $x \log q \in xS \subset S$, so $q^x = \exp(x \log q)$ is well defined. We define the *q-bracket* $[X]_q$ on O_p by

$$[x]_q \overset{\mathrm{df}}{=} \begin{cases} \frac{q^x-1}{q-1} & \text{if } q \neq 1 \\ x & \text{if } q = 1 \end{cases}$$

The q-bracket is an interpolation to O_p of the "q-number" or "q-analog" of $n \in \mathbb{N}$, which is defined by $[n]_q = \frac{q^n-1}{q-1} = 1 + q + q^2 + \cdots + q^{n-1}$. The q-bracket is also the canonical 1-cocycle $[X]_q \in \mathrm{Z}^1(\mathrm{O}_p, \mathrm{O}_p)$ sending 1 to 1, where O_p is viewed as an O_p-module via the action $1 * 1 = q$. If $q \in \mathbb{Z}_p$, then $[X]_q \in \mathrm{Z}^1(\mathbb{Z}_p, \mathbb{Z}_p)$, and then $[X]_q (\mathrm{mod}\, p^n) \in \mathrm{Z}^1(\mathbb{Z}/p^n, \mathbb{Z}/p^n)$.

For reference on basic concepts see the beautiful book [G] by Gouvêa. The study of q-functions for a general variable q tending to 1 is very old, and the study of q-numbers and q-identities goes back at least to Jackson in [J]. In [F] Fray proved p-adic q-analogs of theorems of Legendre, Kummer, and Lucas on q-binomial coefficients. The structure of the space of continuous functions $C(K, \mathbb{Q}_p)$, where K is a local field, was studied by Dieudonné in [D], and Mahler constructed an explicit basis for this space in [M]. In [C] Conrad proved that the q-binomial coefficients form a basis for $C(\mathbb{Z}_p, \mathbb{Z}_p)$. Isometries on $\mathbb{Z}_p$ or on locally compact connected one-dimensional abelian groups have been studied in [A], [B], and [Su].

II. Results.

Proposition 1. *Fix $q \in 1+S$. Then $[X]_q : \mathrm{O}_p \to \mathrm{O}_p$ is a norm-preserving isometry.*

Proof. This is clear for $q = 1$, so assume $q \neq 1$. $[X]_q$ is the composition of analytic isomorphisms

$$\mathrm{O}_p \xrightarrow{\log q} \mathrm{B}(0, |q-1|) \xrightarrow{\exp(-)-1} \mathrm{B}(0, |q-1|) \xrightarrow{\frac{1}{q-1}} \mathrm{O}_p$$

a dilation by $|q-1|$, an isometry, and a contraction by $|q-1|$. Tracing through the maps shows $[X]_q$ preserves the norm.

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Since $[X]_q$ is an isometry of the p-adic unit disk onto itself, the notion of fixed point makes sense. Set

$$f(X, Q-1) = [X]_Q - X$$

For fixed $q \in 1+S$ the set of fixed points of $[X]_q$ is the set of solutions $\{x : f(x, q-1) = 0\}$. We see that $f(X, Q-1)$ is analytic on $\mathrm{O}_p \times S$, since upon rewriting the series we find

$$f(X, Q-1) = \sum_{n=1} \binom{X}{n+1} (Q-1)^n$$

is in $\mathbb{C}_p[[X, Q-1]]$ and converges on $\mathrm{O}_p \times S$.

Notation. We will write $Q - 1 = Y = p^{m_0}U$, so that $q = Q(y) = Q(u) \in 1+S$ for $y \in S$ and $u \in \mathrm{O}_p^*$. Set $A_0(X) = 1$, and for $n > 0$ set

$$A_n(X) = (X-2)(X-3)\cdots(X-(n+1))$$

It is obvious that $f(x, 0) = 0$ for all $x \in \mathrm{O}_p$, and $f(0, q-1) = f(1, q-1) = 0$ for all $q - 1 \in S$. We call these solutions *trivial*, and are led to define

$$M \stackrel{\mathrm{df}}{\equiv} V\left(\frac{[X]_Q - X}{(Q-1)X(X-1)}\right)$$

This set parameterizes the pairs $(x, q-1)$ such that x is a nontrivial fixed point for $(X)_q$.

Proposition 2. *The set M is a submanifold of $\mathrm{O}_p \times S$. If $(x, q-1) \in M$ then there is an analytic function $Q(X)$ in a neighborhood N of x such that $q = Q(x)$, and $(x', Q(x') - 1) \in M$ for all $x' \in N$.*

Proof. Let $Y = Q - 1$ and

$$g(X, Y) = \frac{f(X, Y)}{YX(X-1)}$$

We show dg does not vanish on M by showing $\frac{\partial g}{\partial Y}(x, y) \neq 0$ or all $(x, y) \in M$. Then M is a submanifold of $\mathrm{O}_p \times S$ by [Se, Chapter III, Section 11], and there is an analytic function $Q(x)$ such that $(x', Q(x') - 1) \in M$ in a neighborhood of (x, y) by the p-adic implicit function theorem. Directly from the definition of f,

$$\frac{\partial f}{\partial Y} = \frac{\partial [X]_Q}{\partial Q} = \frac{XQ^{X-1} - [X]_Q}{Q-1}$$

If $(x, y) \in M$, then $(x, y) \in V(f)$, so $[x]_q = x$. Therefore

$$\frac{\partial f}{\partial Y}(x, y) = x[x-1]_q$$

Since $g = f/YX(X-1)$,

$$\frac{\partial g}{\partial Y} = \frac{1}{YX(X-1)}\frac{\partial f}{\partial Y} - \frac{g}{Y}$$

By the power series expression for g we have $g(X, 0) = 1/2$, so in particular if $(x, y) \in M$ then $y \neq 0$, hence $(g/Y)(x, y) = 0$, hence

$$\frac{\partial g}{\partial Y}(x, y) = \frac{1}{yx(x-1)}\frac{\partial f}{\partial Y}(x, y) = \frac{[x-1]_q}{y(x-1)}$$

Expand

$$[X]_q = \sum_{n=1} \frac{(\log q)^n}{(q-1)n!} X^n$$

Therefore $\frac{[X-1]_q}{X-1} = \sum_{n=0} \frac{(\log q)^{n+1}}{(q-1)(n+1)!}(X-1)^n$. This has value $\log q/(q-1)$ at $X = 1$, and if $X \neq 1$ then it is nonzero since $[X-1]_q$ preserves the norm in O_p by Proposition 1. We conclude $\frac{\partial g}{\partial Y}(x, y) \neq 0$ for all $(x, y) \in M$.

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Definitions. Let $\phi_1 : M \to \mathrm{O}_p$ and $\phi_2 : M \to S$ be the projections, and let $M_y = \phi_2^{-1}(y)$ and $M_x = \phi_1^{-1}(x)$ denote the corresponding fibers. We identify M_y with $\phi_1(M_y)$, and M_x with $\phi_2(M_x)$. Thus M_y is the set of nontrivial fixed points of $[X]_q$, where $q = 1 + y$.

Series 1. For any $x \in \mathrm{O}_p$ and $q \in 1 + S$, write

$$\begin{aligned}[X]_q - X &= \sum_{n=0} c_n(x, q)(X - x)^n \\ &= ([x]_q - x) + \left(\frac{q^x \log q}{q-1} - 1\right)(X - x) + \sum_{n=2} \frac{q^x (\log q)^n}{(q-1)n!}(X - x)^n\end{aligned}$$

Proposition 3. *If $p = 2$ then $M = \varnothing$. If $p \neq 2$, then ϕ_2 has degree $p - 2$, and*

$$\phi_2(M) = \{q - 1 \in S : p^{-1/(p-2)} \leq |q - 1| < p^{-1/(p-1)}\}$$

Proof. Fix $q \in 1 + S$. For $x \in \mathrm{O}_p$, let $c_n = c_n(x, q)$ be the coefficient from Series 1. By the p-adic Weierstrass preparation theorem [G, Theorem 6.2.6], the number of zeros of $[X]_q - X$ is $N = \sup\{n : v(c_n) = \inf_m v(c_m)\}$, counting multiplicities. Since $\{0, 1\}$ are both zeros, we know $N \geq 2$, and M_{q-1} has cardinality $N - 2$. For $n \geq 2$, we compute

$$v(c_n) = (n-1)m_0 - \frac{n - s_p(n)}{p-1}$$

where $s_p(n)$ is the sum of the coefficients of the p-adic expansion of n. It is easy to see $v(c_n) > v(c_p)$ whenever $n > p$. If $p = 2$, $v(c_2) = m_0 - 1$. If $p \neq 2$,

$$(*) \qquad v(c_n(q,x)) = \begin{cases} m_0 + nm_0 & \text{if } 2 \leq n \leq p-1 \\ (p-1)m_0 - 1 & \text{if } n = p-2 \end{cases}$$

Thus if $p = 2$ or $m_0 > 1/(p-2)$ we have $N = 2$, hence $M_{q-1} = \varnothing$. If $p \neq 2$ and $m_0 \leq 1/(p-2)$ then $v(c_p) \leq v(c_2)$, hence $N = p$, hence M_{q-1} has cardinality $p-2$, counting multiplicities. We conclude ϕ_2 has degree $p-2$ over $1/(p-1) < m_0 \leq 1/(p-2)$.

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Series 2. Let $Q = 1 + p^{m_0}U$, where $m_0 > 1/(p-1)$. For any $u \in \mathrm{O}_p$,

$$h_{m_0}(x,U) \overset{\mathrm{df}}{=} \frac{(x)_Q - x}{(Q-1)x(x-1)} = \sum_{n=0} d_n(x,u)(U-u)^n$$

where $d_n(x,u) = \sum_{k=n}^{\infty} \binom{k}{n} \frac{A_k(x)}{(k+2)!} p^{km_0} u^{k-n}$. Since $m_0 > 1/(p-1)$, the series converges on O_p^* for all $x \in \mathrm{O}_p$.

Proposition 4. *Suppose $p \neq 2$. Then ϕ_1 has degree $p-2$, and*

$$\phi_1(M) = \{x \in \mathrm{O}_p : A_{p-2}(x) \in S\}$$

If $(x, q-1) \in M$ then $v(A_{p-2}(x)) = 1 - (p-2)m_0$.

Proof. Since $p \neq 2$, $M \neq \varnothing$ and $1/(p-1) < m_0 \leq 1/(p-2)$ by Proposition 3. Write $d_n = d(x,0)$ for the coefficient in Series 2. Then

$$v(d_n) = v(A_n(x)) + nm_0 + \frac{s_p(n+2) - (n+2)}{p-1}$$

and from this we read off

$$(**) \qquad v(d_n(x,0)) = \begin{cases} v(A_n(x)) + nm_0 & \text{if } 0 \leq n \leq p-3 \\ v(A_{p-2}(x)) + (p-2)m_0 - 1 & \text{if } n = p-2 \end{cases}$$

If $n > p-2$, we easily compute $v(d_n) - v(d_{p-2}) > 0$ using $m_0 > 1/(p-1)$. It follows that the Weierstrass polynomial has nonzero degree if and only if $v(d_{p-2}) \leq v(d_0)$, i.e., $v(A_{p-2}(x)) \leq 1-(p-2)m_0$, and then the degree is $p-2$. Since m_0 may assume any value greater than $1/(p-1)$, for a given x this holds for all m_0 if and only if $v(A_{p-2}(x)) < 1/(p-1)$, i.e., $A_{p-2}(x) \in S$. If x is a nontrivial fixed point for $[X]_q$ then the first segment of the Newton polygon must be horizontal to ensure the solution $U = u$ is a unit. Thus $m_0 = (1 - v(A_{p-2}(x)))/(p-2)$.

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Proposition 5. *Suppose $M_{q-1} \neq \varnothing$, so that* $\mathrm{Card}(M_{q-1}) \leq p-2$, *and write $\overline{M}_{q-1}$ for the set of residues.*

a. *If $m_0 < 1/(p-2)$, $\overline{M}_{q-1} = \{2, \dots, p-1\}$, and* $\mathrm{Card}(M_{q-1}) = p-2$.
b. *If $m_0 = 1/(p-2)$, $\overline{M}_{q-1} \cap \{2, \dots, p-1\} = \varnothing$, and* $\mathrm{Card}(M_{q-1}) \geq p-3$.

Proof. Since $1/(p-1 < m_0 \leq 1/(p-2)$, the Weierstrass polynomial for $[X]_q - X$ in Series 1 has degree p by Proposition 3, and each $[X]_q$ has a fixed point. We have $v(A_{p-2}(x)) = 0$ if and only if $\bar{x} \notin \{2, \dots, p-1\}$ if and only if $m_0 = 1/(p-2)$ by Proposition 4. This proves all but the cardinality statements.

If $m_0 = 1/(p-2)$ then $v(c_2(x,q)) = v(c_p(x,q)) = m_0$ by $(*)$, so there are at most two zeros with residue $\bar{x}$ by $(*)$. Suppose $[X]_q$ has fixed points x and x', such that $\bar{x} \neq \bar{x}'$. We compute $c_1(x',q) - c_1(x,q) = ([x']_q - [x]_q)\log q$, and $[x']_q - [x]_q$ is a unit by Proposition 1. Therefore if $v(c_1(x,q)) > m_0$ then $v(c_1(x',q) - c_1(x,q)) = m_0$, hence $v(c_1(x',q)) = m_0$. Thus if there are two points in M_{q-1} with the same residue, then the remaining residues are distinct, hence $\mathrm{Card}(M_{q-1}) \geq p-3$.

If $m_0 < 1/(p-2)$ then $v(c_2(x,q)) > v(c_p(x,q))$ by $(*)$, so since not every fixed point for $[X]_q$ has the same residue we must have $v(c_1(x,q)) = v(c_p(x,q))$, hence there is at most one $x \in M_{q-1}$ with any given residue, and $\mathrm{Card}(M_{q-1}) = p-2$.

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Remark. By Proposition 4 and Proposition 5, we compute

$$\phi_1(M) = \bigcup_{\bar{a} \notin \{\bar{2}, \dots, \bar{p}-\bar{1}\}} \mathrm{B}(a,1) \ \cup \bigcup_{a \in \{2, \dots, p-1\}} \mathrm{B}(a,1) - \bar{\mathrm{B}}(a, p^{-1/(p-1)})$$

where the left union corresponds to $v(q-1) = 1/(p-2)$, the right union to $v(q-1) < 1/(p-2)$, and $a \in \mathrm{O}_p$. Note no rational integer not congruent to 0 or 1 $(\mathrm{mod}\, p)$ may be a fixed point for any $[X]_q$.

Proposition 6. *Suppose $(x, q-1) \in M$. Then* $\mathrm{Card}(M_x) = p-2$, *and each $u = p^{-m_0}(q-1)$ has a different residue. If $x' \in \mathrm{B}(x, |A_{p-2}(x)|)$ then there exists a unique $q' \in \mathrm{B}(q, |q-1|)$ such that $(x', q'-1) \in M$, and we have a map*

$$Q(X) : \mathrm{B}(x, |A_{p-2}(x)|) \to \mathrm{B}(q, |q-1|)$$

such that $|q'-q| = |([x']_q - x')/x'(x'-1)|$.

Proof. Since x is a nontrivial fixed point, $0 \leq v(A_{p-2}(x)) < 1/(p-1)$ and M_x has cardinality $p-2$, counting multiplicities, by Proposition 4. By Series 2,

$$h_{m_0}(x, U) = \sum_{n=0} d_n(x,u)(U-u)^n$$

where $d_n(x,u) = \sum_{k=n} \binom{k}{n} \frac{A_k(x)}{(k+2)!} p^{km_0} u^{k-n}$. Thus $d_0(x,u) = h_{m_0}(x,u) = 0$, and using $(**)$ we compute for $1 \leq n \leq p-2$, $v(d_n(x,u)) = 0$. Thus the Newton polygon contains the points $(n, v(d_n(x,u))) = (0, \infty), (1, 0), \dots, (p-2, 0)$, which shows the $p-2$ roots u have distinct residues.

We show $v(d_0(x', u)) > 0$ for $x' \in \mathrm{B}(x, |A_{p-2}(x)|)$. As $d_0(x, u) = 0$, it is equivalent to show $v(A_{p-2}(x') - A_{p-2}(x)) > 1 - (p-2)m_0 = v(A_{p-2}(x)$. Since $A_{p-2}(X) = 1 + X + \cdots + X^{p-2} (\bmod p)$, this is equivalent to

$$v((x' - x) + ((x')^2 - x^2) + \cdots + ((x')^{p-2} - x^{p-2})) > v(A_{p-2}(x)$$

and the condition holds since $x' \in \mathrm{B}(x, |A_{p-2}(x)|$. Next,

$$d_1(x', u)) = \sum_{k=1} k \frac{A_k(x')}{(k+2)!} p^{km_0} u^{k-1} = \frac{A_1(x')}{3!} p^{m_0} + \cdots + (p-2) \frac{A_{p-2}(x')}{(p-1)!} p^{(p-2)m_0 - 1} u^{p-3} + \cdots$$

and $v(d_1(x', u)) = v(A_{p-2}(x') p^{(p-2)m_0 - 1})$. As $x' \in \mathrm{B}(x, |A_{p-2}(x)|)$, $v(A_{p-2}(x')) = v(A_{p-2}(x))$, so $v(d_1(x', u)) = v(d_1(x, u) = 0$. By the Newton polygon, for each $x' \in \mathrm{B}(x, |A_{p-2}(x)|)$ there is a unique $q' = 1 + p^{m_0} u'$, such that x' is a nontrivial fixed point for $[X]_{q'}$, and $v(u' - u) = v(d_0(x', u)) - v(d_1(x', u)) = v(d_0(x', u))$, hence $v(q' - q) = v(h_{m_0}(x', u)) + m_0$. The result follows since $h_{m_0}(x', u) = ([x']_q - x')/((q-1)x'(x'-1))$.

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Proposition 7. *Suppose $(x, q-1) \in M$. If x has multiplicity one in M_{q-1}, then $|Q(x') - q| \sim |x' - x|$ for x' sufficiently near x. Otherwise $|Q(x') - q| \sim |x' - x|^2$ for x' sufficiently near x.*

Proof. By Series 1, x has multiplicity one in M_{q-1} if and only if $q^x \log q \neq q - 1$ or $x \in \{0, 1\}$. We expand $([X]_q - X)/((q-1)X(X-1))$ around x. For $x \neq 0, 1$,

$$\frac{1}{X(X-1)} = \sum_{n=0} a_n (X - x)^n$$

where $a_n = (-1)^{n+1}(x^{-(n+1)} - (x-1)^{-(n+1)})$. Then

$$g(X, q) = \frac{[X]_q - X}{X(X-1)} = \sum_{n=0} b_n(x, q)(X - x)^n$$

where $b_n = \sum_{i+j=n} a_i c_j$, where $c_j = c_j(x, q-1)$ is the coefficient from Series 1. Since $(x, q-1) \in M$, $c_0 = 0$, and we obtain $b_0 = 0$, $b_1 = a_0 c_1 = (1 - q + q^x \log q)/((q-1)x(x-1))$, and if $b_1 = 0$ then $b_2 = a_0 c_2 = q^x (\log q)^2/((q-1)x(x-1))$. Whenever $b_1 \neq 0$, for x' sufficiently near x we have $v(q' - q) = v(x' - x) + v(b_1(x, q))$ by Proposition 6. If $b_1 = 0$, i.e., $q^x \log q = q - 1$, then $b_2 \neq 0$, and then for x' sufficiently near x we have $v(q' - q) = 2v(x' - x) + v(b_2(x, q))$.

For $x = 0$ we use $1/(X-1) = \sum_{n=0} -X^n$, and compute $g(X, q) = \sum_{n=0} b_n X^n$ where $b_n = -(c_1 + \cdots + c_{n+1})$. Since $(x, q-1) \in M$, x already has multiplicity two, so $c_1 = 0$, hence $b_0 = 0$, and since $c_2 \neq 0$ is guaranteed, for x' sufficiently near $x = 0$ we have $|q' - q| \sim |x' - x|$. For $x = 1$ we use $1/X = \sum_{n=0} (-1)^n (X-1)^n$, and $g(X, q) = \sum_{n=0} b_n (X-1)^n$ where $b_n = c_1 - c_2 + c_3 - \cdots + (-1)^n c_{n+1}$, and we obtain the same result, for x' sufficiently near $x = 1$.

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Proposition 8. *Suppose* $(x, q-1) \in M$, $x' \in \mathrm{B}(x, |A_{p-2}(x)|)$, *and* $q' = Q(x')$. *If* $p = 3$ *then locally* $Q(X) : \mathrm{B}(x, |x-2|) \to \mathrm{B}(q, |q-1|)$ *is an analytic surjection satisfying* $|q'-q| = |x'-x|p^{1-2m_0}$. *If* x *has multiplicity two in* M_{q-1} *then* $v(A^{(1)}_{p-2}(x)) = 1-(p-3)m_0$.

Proof. Write $q-1 = p^{m_0}u$ and $q'-1 = p^{m_0}u'$. We have $v(u'-u) = v(h_{m_0}(x', u))$ by Proposition 6. Since $h_{m_0}(x, u) = 0$,

$$h_{m_0}(X, u) = h_{m_0}(X, u) - h_{m_0}(x, u) = \sum_{n=1} \frac{A_n(X) - A_n(x)}{(n+2)!} p^{nm_0} u^n$$

If $p = 3$ then $A_{p-2}(x) = A_1(x) = x-2$, and it is easy to see $v(h_{m_0}(x', u)) = v(x'-x) + m_0 - 1$. Thus $v(q'-q) = v(x'-x) + 2m_0 - 1$, and the map $Q(X) : \mathrm{B}(x, |x-2|) \to \mathrm{B}(q, |q-1|)$, which is analytic by Proposition 2, is surjective: If $q' \in \mathrm{B}(q, |q-1|)$ then $q' = \phi_2(x')$ for some x' by Proposition 3, and $v(x'-x) = v(q'-q) + 1 - 2m_0 > 1 - m_0 = v(x-2)$, so $x' \in \mathrm{B}(x, |x-2|)$.

Assume $p \neq 3$, so $p - 2 \neq 1$. Compute

$$A_{p-2}(x') - A_{p-2}(x) = A^{(1)}_{p-2}(x)(x'-x) + \frac{1}{2!} A^{(2)}_{p-2}(x)(x'-x)^2 + \cdots + \frac{1}{(p-1)!} A^{(p-1)}_{p-2}(x)(x'-x)^{p-1}$$

If x has multiplicity two in M_{q-1} then $|q'-q| \sim |x'-x|^2$ by Proposition 7, and since $p \neq 3$ this can only happen if $v(A^{(1)}_{p-2}(x)) + (p-2)m_0 - 1 = m_0$, as claimed.

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Remark. Since $A_{p-2}(X) = 1 + X + \cdots + X^{p-2} (\operatorname{mod} p)$, we have

$$A^{(1)}_{p-2}(X) = 1 + 2X + \cdots + (p-2)X^{p-3} (\operatorname{mod} p)$$

Thus if $p \neq 3$ and $\bar{x} \in \mathbb{F}_p$ then since $A^{(1)}_{p-2}(x) \neq 0$, x is not a double point for $[X]_q$.

Proposition 9. *Let* $M(\mathbb{Z}_p) = \{(x, q-1) \in M : x, q \in \mathbb{Z}_p\}$. *Then*

$$M(\mathbb{Z}_p) \neq \varnothing \iff p = 3$$

The image of ϕ_1 *is* $\phi_1(M(\mathbb{Z}_3)) = \mathrm{B}(1,1) \cup \mathrm{B}(0,1)$, *and we have an analytic surjection*

$$\begin{aligned} Q(X) : \mathrm{B}(1,1) &\longrightarrow \mathrm{B}(4, 3^{-1}) \\ \mathrm{B}(0,1) &\longrightarrow \mathrm{B}(7, 3^{-1}) \end{aligned}$$

with $|q'-q| = |x'-x|/3$ *about each* $x \notin \{0,1\}$. *Thus if* $q = 1 + 3u$ *for* $u \in \mathbb{Z}_3^*$, *the map* $U(X)$ *taking* x' *to* u' *yields isometries* $\mathrm{B}(1,1) \xrightarrow{\sim} \mathrm{B}(1,1)$ *and* $\mathrm{B}(0,1) \xrightarrow{\sim} \mathrm{B}(2,1)$.

Proof. By Proposition 3, $M \neq \varnothing$ if and only if $1/(p-1) < v(q-1) \leq 1/(p-2)$, so we have the first statement. Assume $p = 3$. By Proposition 4, $\phi_1(M) \cap \mathbb{Z}_3 = \{x : x \neq 2 (\operatorname{mod} 3)\}$, and by Proposition 3, $\phi_2(M) \cap \mathbb{Z}_3 = \{q - 1 : |q-1| = 3^{-1}\}$. Since the Weierstrass polynomial for Series 2 has degree one, we see that $x \in \mathbb{Z}_3$ if and only if $q \in \mathbb{Z}_3$, so these sets are $\phi_1(M(\mathbb{Z}_3))$ and $\phi_2(M(\mathbb{Z}_3))$, respectively. Locally the map $Q(X)$ takes $\mathrm{B}(x,1)$ onto $\mathrm{B}(q, 3^{-1})$ by Proposition 8, and is a contraction by $1/3$. By sheer luck we find the nontrivial fixed point $-1/2$ for $q = 4$, and since $-1/2$ has residue 1, we conclude that $Q(X)$ takes $\mathrm{B}(1,1)$ onto $\mathrm{B}(4, 3^{-1})$ and $\mathrm{B}(0,1)$ onto $\mathrm{B}(7, 3^{-1})$. The last statement is trivial.

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